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LETTER TO THE EDITOR

On colour superalgebras in parasupersymmetric quantum mechanics

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Abstract. The simplest Lie parasuperalgebra Psqm(2) is identified, as an example, with a C(2, S)-colour superalgebra through simple connections between trilinear and bilinear structure relations but by using Green–Cusson ansätze developed in parastatistics. Some general properties and conclusions are then obtained and illustrated.

Topics such as parastatistics [1, 2] and supersymmetry [3, 4] have recently received much attention within the so-called parasupersymmetric quantum mechanics [5–7]. The first non-trivial considerations correspond to the superposition of ordinary bosons with order $p=2$ parafermions. Such a context leads in particular to new mathematical structures which have been called 'Lie parasuperalgebras' [8, 9]. They are characterized by a Lie bracket appearing as double commutators associated with *trilinear* operator products and leading to new (diagonalized) parasuperHamiltonians and to (conserved) parasupercharges.

The simplest Lie parasuperalgebra [8] called Psqm(2) by reference to the original Witten work in supersymmetric quantum mechanics [3], is given through the following structure relations:

$$\begin{aligned} &[Q,[Q^\dagger, Q]]=2QH_{\mathrm{PSS}} \qquad [Q^\dagger,[Q, Q^\dagger]]=2Q^\dagger H_{\mathrm{PSS}} \\ &[H_{\mathrm{PSS}}, Q]=0 \qquad [H_{\mathrm{PSS}}, Q^\dagger]=0. \end{aligned} \tag{1}$$

The parasupersymmetric Hamiltonian H_{PSS} and the parasupercharge Q respectively play the role of even ($\mathscr{E}$) and odd ($\mathscr{O}$) generators in order to recover as a particular context the supersymmetric case: this is easily understood by noticing the following identity connecting commutators and anticommutators:

$$[A,[B,C]]=\{\{A,B\},C\}-\{\{A,C\},B\} \tag{2}$$

and by recalling that a Lie superalgebra contains structure relations associated with a Z_2 graduation characterized by

$$[\mathscr{E},\mathscr{E}]\sim\mathscr{E} \qquad [\mathscr{E},\mathscr{O}]\sim\mathscr{O} \qquad \{\mathscr{O},\mathscr{O}\}\sim\mathscr{E}. \tag{3}$$

Here we want to show that the above Lie parasuperalgebras in fact appear as specific generalized algebras (constructed by Rittenberg and Wyler [10]) which have been called 'colour superalgebras'. In order to establish such a result we need to work *first* with Green–Cusson ansätze [1, 11] for the concerned parafermions associated

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with oscillator-like considerations characterized by an angular frequency (ω) taken here equal to one. Let us choose the following parafermionic realizations:

$$B = b_1\sigma_1 + b_2\sigma_2 \qquad B^\dagger = b_1^\dagger\sigma_1 + b_2^\dagger\sigma_2 \tag{4}$$

expressed in terms of Hermitian Pauli matrices and *ordinary* fermionic annihilation and creation operators b_i, $b_i^\dagger$ ($i = 1, 2$) satisfying the usual anticommutation relations

$$\{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0 \qquad \{b_i, b_j^\dagger\} = \delta_{ij} I_2. \tag{5}$$

These four fermionic operators are easily seen as Clifford matrices belonging to a Cl_4-algebra [12]. for example, they are given by the direct products:

$$\begin{aligned} &b_1 = \sigma_3 \otimes \sigma_- \qquad b_1^\dagger = \sigma_3 \otimes \sigma_+ \qquad b_2 = \sigma_- \otimes I_2 \\ &b_2^\dagger = \sigma_+ \otimes I_2 \qquad \sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2). \end{aligned} \tag{6}$$

Within such Green–Cusson ansätze (4), our parasupercharge Q appears in the form

$$Q = \frac{\mathrm{i}}{2}(a^\dagger B^\dagger B^2 - aB^2 B^\dagger) = Q_1 + Q_2 = Q_1'\sigma_1 + Q_2'\sigma_2 \tag{7}$$

where

$$Q_1' = \mathrm{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a^\dagger & \cdot & \cdot & \cdot \\ \cdot & a & \cdot & \cdot \end{pmatrix} \qquad Q_2' = \mathrm{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ a^\dagger & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & -a & \cdot & \cdot \end{pmatrix} \tag{8}$$

while the *ordinary* bosonic operators $a, a^\dagger$ are defined as usual. By noticing that

$$\{Q_i', Q_j'\} = \{Q_i'^\dagger, Q_j'^\dagger\} = 0 \qquad \{Q_i', Q_j'^\dagger\} = \delta_{ij} H' \tag{9a}$$

where

$$H' = \mathrm{diag}(aa^\dagger, a^\dagger a, a^\dagger a, aa^\dagger) \tag{9b}$$

we immediately get

$$\{Q_j, Q_j\} = \{Q_j^\dagger, Q_j^\dagger\} = 0 \qquad \{Q_j, Q_j^\dagger\} = H_{\mathrm{PSS}} \tag{10a}$$

$$[H_{\mathrm{PSS}}, Q_j] = 0 \qquad [Q_j, Q_k] = [Q_j, Q_k^\dagger] = [Q_j^\dagger, Q_k^\dagger] = 0 \qquad (j \neq k) \tag{10b}$$

$$H_{\mathrm{PSS}} = H' \otimes I_2. \tag{11}$$

As is easily verified, the trilinear structure relations (1) are recovered through the bilinear ones (10) with the definition (7) of the parasupercharge.

Let us now show as a *second* step that H, Q_1, $Q_1^\dagger$, Q_2 and $Q_2^\dagger$ generate a colour superalgebra belonging to the so-called $C(2, S)$-family [10]. If we refer to two-dimensional grading vectors $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} \equiv (\beta_1, \beta_2)$ whose components are integers modulo 2, the symmetry property of the corresponding generalized lie bracket is such that for $C(2, S)$-colour superalgebras we have

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \alpha_1\beta_1 + \alpha_2\beta_2. \tag{12}$$

There are in that case *four* classes of generalized generators that are called $X_{(0,0)k}$, $X_{(0,1)k}$, $X_{(1,0)k}$, $X_{(1,1)k}$ with the following structure relations

$$\begin{aligned}
&[X_{(0,0)k}, X_{(0,0)m}] = C^{(0,0)n}_{(0,0)k;(0,0)m} X_{(0,0)n} \\
&[X_{(0,0)k}, X_{(0,1)m}] = C^{(0,1)n}_{(0,0)k;(0,1)m} X_{(0,1)n} \\
&[X_{(0,0)k}, X_{(1,0)m}] = C^{(1,0)n}_{(0,0)k;(1,0)m} X_{(1,0)n} \\
&[X_{(0,0)k}, X_{(1,1)m}] = C^{(1,1)n}_{(0,0)k;(1,1)n} X_{(1,1)n}
\end{aligned} \qquad (13a)$$

$$\begin{aligned}
&[X_{(1,1)k}, X_{(1,1)m}] = C^{(0,0)n}_{(1,1)k;(1,1)m} X_{(0,0)n} \\
&\{X_{(0,1)k}, X_{(0,1)m}\} = C^{(0,0)n}_{(0,1)k;(0,1)m} X_{(0,0)n} \\
&\{X_{(1,0)k}, X_{(1,0)m}\} = C^{(0,0)n}_{(1,0)k;(1,0)m} X_{(0,0)n}
\end{aligned} \qquad (13b)$$

$$\begin{aligned}
&[X_{(0,1)k}, X_{(1,0)m}] = C^{(1,1)n}_{(0,1)k;(1,0)m} X_{(1,1)n} \\
&\{X_{(0,1)k}, X_{(1,1)m}\} = C^{(1,0)n}_{(0,1)k;(1,1)m} X_{(1,0)n} \\
&\{X_{(1,0)k}, X_{(1,1)m}\} = C^{(0,1)n}_{(1,0)k;(1,1)m} X_{(0,1)n}.
\end{aligned} \qquad (13c)$$

We notice that the diagonal products involve commutators as well as anticommutators; we are thus dealing here with a colour superalgebra. Such a set of structure relations does coincide with our relations (10) by identifying the following operators:

$$\begin{aligned}
&X_{(0,0)} \equiv H_{\mathrm{PSS}} \qquad X_{(0,1),1} \equiv Q_1 \qquad X_{(0,1),2} \equiv Q_1^\dagger \\
&X_{(1,0)1} \equiv Q_2 \qquad X_{(1,0)2} \equiv Q_2^\dagger \qquad X_{(1,1)} \equiv 0.
\end{aligned} \qquad (14)$$

The corresponding colour superalgebra is also isomorphic to the algebra of creation and annihilation operators of one parafermion of order two realized through the Green ansatz [1] when we relate the Hamiltonian to the identity operator. Let us also point out that it is easy to get the expected Lie superalgebra corresponding to this Psqm(2) according to the (second) theorem established by Rittenberg and Wyler [10] for colour superalgebras. Indeed here we find that this Lie superalgebra C(1, S) is Sqm(4) characterized by the structure relations

$$\{q_b^a, q_d^c\} = 2\delta^{ac}\delta_{bd}h \qquad (a, b, c, d = 1, 2) \qquad (15)$$

where the four *Hermitian* generators q_b^a are defined by

$$q_1^1 = q_1 + q_1^\dagger \qquad q_1^2 = \mathrm{i}(q_1 - q_1^\dagger) \qquad q_2^1 = q_2 + q_2^\dagger \qquad q_2^2 = \mathrm{i}(q_2 - q_2^\dagger) \qquad (16)$$

with

$$q_1 = Q_1 \otimes \sigma_1 \qquad q_2 = Q_2 \otimes \sigma_2 \qquad h = H_{\mathrm{PSS}} \otimes \sigma_0. \qquad (17)$$

This five-dimensional superalgebra is here represented by (16×16) matrices as appears clearly from (7), (8) and (17).

The above results and properties can now be extended to other Lie parasuperalgebras. In order to facilitate their understanding, let us point out some facts already perceptible inside the above simplest example. The four C(2,S)-classes of generators are issued from the $Z_2 \oplus Z_2$ graduation introduced by Rittenberg and Wyler [10]. To ask for even and odd generators (as it is the case in Lie superalgebras [13]) corresponds to only one Z_2 graduation. This last case is easily connected to the previous considerations by requiring that

$$\{\mathscr{E}\} \equiv \{X_{(0,0)k}, X_{(1,1)k}\} \qquad \{\mathscr{O}\} \equiv \{X_{(0,1)k}, X_{(1,0)k}\}. \qquad (18)$$

We thus say that an *even* generator is characterized by the sum of the grading vector components equal to *zero* (mod 2) and an *odd* one by the sum equal to *one* (mod 2). This explains the above identifications (18) and correlatively (14). Moreover, inside the class of *even* generators, we identify our parasuperoperators with those contained in a subset of the class $\{X_{(0,0)k}\}$ due to the following remark: according to the even and odd characters given by (18), we notice that equations (13*a*) *and* (13*b*) are compatible with equations (1) but equations (13*c*) are *not.* By noticing that these last abnormal equations (13*c*) are *due to* the presence of the even generators $X_{(1,1)k}$, we need to identify our even parasuperoperators with the $X_{(0,0)k}$ only (see also (14) where we have annihilated $X_{(1,1)}$) when these even parasuperoperators are playing a role in the supercontext. This remark partly points out also that parasuperalgebras or colour superalgebras are structures *larger than* Lie superalgebras due precisely to the inclusion of new structure relations of the type

$$[\mathcal{O}, \mathcal{O}] \sim \mathcal{E} \qquad \textit{and} \qquad \{\mathcal{O}, \mathcal{E}\} \sim \mathcal{O} \tag{19}$$

which are not included in equations (3). In the future such relations (19) will effectively help us to construct and to identify the new *even* operators which have to be included in the class $\{X_{(1,1)k}\}$ for other larger parasuperstructures such as the ones discussed hereafter.

The *third* step of our discussion will concern some general properties that we can exploit for analysing Lie parasuperalgebras in terms of Lie colour superalgebras of C(2, S)-type or conversely. Let us immediately state that the bilinear relations (13) of a C(2,S)-colour superalgebra always give sense to the *whole* set of typical trilinear structure relations (already present in equations (1)) given by

$$[\mathcal{O}, [\mathcal{O}, \mathcal{O}]] \sim [\mathcal{O}, \mathcal{E}] \tag{20}$$

of the associated parasuperalgebra [8]. Indeed such a property can explicitly be established by considering the whole set of odd generators $\{X_{(0,1)}, X_{(1,0)}\}$ and their double commutators:

$$[X_{(0,1)j}, [X_{(0,1)k}, X_{(0,1)m}]] \qquad [X_{(1,0)j}, [X_{(1,0)k}, X_{(1,0)m}]] \tag{21a}$$

$$[X_{(0,1)j}, [X_{(0,1)k}, X_{(1,0)m}]] \qquad [X_{(1,0)j}, [X_{(0,1)k}, X_{(1,0)m}]] \tag{21b}$$

$$[X_{(0,1)j}, [X_{(1,0)k}, X_{(1,0)m}]] \qquad [X_{(1,0)j}, [X_{(0,1)k}, X_{(0,1)m}]] \tag{21c}$$

where we notice that only the double commutators (21*a*) and (21*b*) are linearly independent when we take account of Jacobi identities. Consequently our statement says that to each C(2, S)-colour superalgebra always corresponds a definite parasuperalgebra. The converse proposition is *not* true due to the particular peoperties of the *even* generators belonging to the class $\{X_{(1,1)}\}$. In fact such a class can be *empty* (see our previous first example Psqm(2) with the identifications (14)); it can also be fulfilled by a finite *or* an infinite number of generators as it will be the case in our forthcoming examples Psqm(4) (and psh(4/2)) *or* Pspl(2/2) respectively. These properties can be understood by noticing that from a C(2, S)-colour superalgebra given by the structure relations (13), only the double commutators (21*b*) are finally expressed as functions of the even generators $X_{(1,1)}$ inside double *products* with odd ones belonging to the classes $\{X_{(1,0)}\}$ or $\{X_{(0,1)}\}$. Moreover through the relations concerning these products, it is easy to determine if the structure closes (does not close) leading to a finite (infinite) set of generators $X_{(1,1)}$. In the infinite case, no colour superalgebra does correspond to the parasuperalgebra (see the following example of Pspl(2/2)). Such considerations show that parasuperalgebras are still structures more general than C(2,S)-colour superalgebras.

Let us now come back on some *specific* examples of structures corresponding to already known parasuperalgebras [8, 9]. As an immediate extension of Psqm(2), we have constructed [8] the larger parasuperalgebra Psqm(4) characterized by four parasupercharges Q, $Q^\dagger$, P and $P^\dagger$ besides the parasuperHamiltonian H_{PSS}. Their double commutators (as structure relations) having already been given elsewhere [8], let us only identify for brevity here the corresponding classes of the associated C(2, S)-colour superalgebra. Besides the expressions (7) and (8) we have now

$$P = -\tfrac{1}{2}\mathrm{i}(a^\dagger B^\dagger B^2 + aB^2B^\dagger) = P_1 + P_2 = P_1'\sigma_1 + P_2'\sigma_2 \tag{22}$$

where

$$p_1' = \mathrm{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -a^\dagger & \cdot & \cdot & \cdot \\ \cdot & a & \cdot & \cdot \end{pmatrix} \qquad P_2' = \mathrm{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ -a^\dagger & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -a & \cdot \end{pmatrix} \tag{23}$$

and we notice the appearance of a new even generator—let us call it G—issued from relations of type (19) in the bilinear context. We effectively get

$$[Q_1, P_1^\dagger] = [Q_2, P_2^\dagger] = G \qquad [H_{\mathrm{PSS}}, G] = 0 = \{G, Q_1\} = \ldots = 0 \tag{24}$$

where

$$G = G' \otimes I_2 \qquad G' = \begin{pmatrix} aa^\dagger & \cdot & \cdot & \cdot \\ \cdot & -a^\dagger a & \cdot & \cdot \\ \cdot & \cdot & -a^\dagger a & \cdot \\ \cdot & \cdot & \cdot & aa^\dagger \end{pmatrix}. \tag{25}$$

The corresponding identifications are then simply (compare with (14)):

$$\begin{array}{llll} X_{(0,0)} \equiv H_{\mathrm{PSS}} & X_{(1,1)} \equiv G & & \\ X_{(0,1)1} \equiv Q_1 & X_{(0,1)2} \equiv Q_1^\dagger & X_{(0,1)3} \equiv P_2 & X_{(0,1)4} \equiv P_2^\dagger \\ X_{(1,0)1} \equiv Q_2 & X_{(1,0)2} \equiv Q_2^\dagger & X_{(1,0)3} \equiv P_1 & X_{(1,0)4} \equiv P_1^\dagger \end{array} \tag{26}$$

where we only notice that the class $X_{(1,1)}$ is not empty.

Another parasuperalgebra which has already been pointed out with a *physical* interest is the parasupergeneralization of the Heisenberg Lie algebra: we have called this structure Psh(4/2) [9] generated by four odd parasupercharges $T_\pm$, $U_\pm$ and two even Heisenberg operators $P_\pm$. In terms of the preceding matrices and operators, we have (besides the unit matrix)

$$\begin{array}{ll} P_+ = [\exp(-\mathrm{i}t)]a^\dagger & P_- = [\exp(\mathrm{i}t)]a \\ T_+ = [\exp(-\mathrm{i}t)]B^2B^\dagger & T_- = [\exp(\mathrm{i}t)]BB^{\dagger 2} \\ U_+ = [\exp(-\mathrm{i}t)]B^\dagger B^2 & U_- = [\exp(-\mathrm{i}t)]B^{\dagger 2}B \end{array} \tag{27}$$

and it is possible to show that there is an associated C(2,S)-colour superalgebra closing in accordance with equations (13). In fact we find here that this colour superalgebra contains 21 generators falling in the above four classes as follows:

$$\begin{array}{ll} \{X_{(0,0)}\} \equiv \{I, P_\pm, X_1, X_2\} & \\ \{X_{(0,1)}\} \equiv \{T_1, T_1^\dagger, U_1, U_1^\dagger\} & \{X_{(1,0)}\} \equiv \{T_2, T_2^\dagger, U_2, U_2^\dagger\} \\ \{X_{(1,1)}\} \equiv \{X_3, X_4, X_5, X_6\} & \end{array} \tag{28}$$

where, in particular, we quote

$$X_1 = \sigma_3 \otimes I_2 \otimes I_2 \qquad X_2 = I_2 \otimes \sigma_3 \otimes I_2 .$$

We simply point out the appearance of *six* new even generators four of whom belong to the (once again) non-empty class $X_{(1,1)}$.

As a last example, let us consider the parasuperalgebra Pspl(2/2) generated by seven even and eight odd operators and such that

$$\mathrm{Pspl}(2/2) \supset \mathrm{osp}(2/2) \tag{29}$$

has interesting information on the inclusion of supersymmetries inside parasupersymmetries [9]. This elaborate example is the simplest one that we have considered showing that the class $\{X_{(1,1)}\}$ contains an *infinite* number of generators when the associated colour superalgebra is searched. Let us only mention here that with the initial parasupercharges quoted as [9] Q, $Q^\dagger$, P, $P^\dagger$, R, $R^\dagger$, S, $S^\dagger$ we associate here 16 odd generators easily distributed in the two classes $\{X_{(0,1)}\}$ and $\{X_{(1,0)}\}$. Then, by starting with the definitions

$$R = -\tfrac{1}{2}\mathrm{i}\, aB^\dagger B^2 \qquad S = -\tfrac{1}{2}\mathrm{i}\, a^\dagger B^2 B^\dagger \tag{30}$$

it is not too complicated to show that the corresponding odd components lead by bilinear products to new even generators which, through equations (19) for example, give new odd ones and so on. We are thus dealing here with an infinite-dimensional structure which is not a previously defined $\mathrm{C}(2, S)$-colour superalgebra.

Let us end this letter with a few remarks. Due to the simple relation [9] between the triple products defined by Durand and Vinet [14] and by us, it is easily verified that the Durand–Vinet structures can *also* be related to $\mathrm{C}(2, S)$-colour superalgebras if we adopt realizations through our Green–Cusson matrices. If all the above considerations mainly refer to the *standard* supersymmetrization procedure [3, 9], they can also be extended to parasuperalgebras found from the *spin–orbit coupling* supersymmetrization procedure [9, 15–17]. In fact, we just want to mention here that the Ui–Takeda developments [16] at the *supersymmetric* level have never been (to our knowledge) questioned in connection with colour structures. Through our analysis, we learned that Ui and Takeda have also used the Green–Cusson ansätze but for a paraboson of order $p=3$. It is then easy to identify the Ui–Takeda structures with $\mathrm{C}(2, A)$-colour algebras characterized by the (antisymmetric) property

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \alpha_1\beta_2 - \alpha_2\beta_1 \tag{31}$$

replacing the property (12) for $\mathrm{C}(2, S)$-colour superalgebras.

We thank V Rittenberg for drawing our attention to his references [10] on generalized superalgebras.

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